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BI-HAMILTONIAN SYSTEMS ON THE DUAL OF THE LIE ALGEBRA OF VECTOR FIELDS OF THE CIRCLE AND PERIODIC SHALLOW WATER EQUATIONS

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Abstract. This paper is a survey article on bi-Hamiltonian systems on the dual of the Lie algebra of vector fields on the circle. We investigate the special case where one of the structures is the canonical Lie-Poisson structure and the second one is constant. These structures called *affine* or *modified Lie-Poisson structures* are involved in the integrability of certain Euler equations that arise as models of shallow water waves.

1. Introduction

In the last forty years or so, the Korteweg-de Vries equation has received much attention in the mathematical physics literature. Some significant contributions were made in particular by Gardner, Green, Kruskal, Miura (see [46] for a complete bibliography and a historical review). It is through these studies, that emerged the *theory of solitons* as well as the *inverse scattering method*.

One remarkable property of Korteweg-de Vries equation, highlighted at this occasion, is the existence of an infinite number of first integrals. The mechanism, by which these conserved quantities were generated, is at the origin of an algorithm called the *Lenard recursion scheme* or *bi-Hamiltonian formalism* [18, 36]. It is representative of infinite-dimensional systems known as *formally integrable*, in reminiscence of finite-dimensional, classical integrable systems (in the sense of Liouville). Other examples of bi-Hamiltonian systems are the Camassa-Holm equation [16, 4, 6, 14, 21] and the Burgers equation.

One common feature of all these systems is that they can be described as the geodesic flow of some right-invariant metric on the diffeomorphism group of the circle or on a central real extension of it, the Virasoro group. Each left (or right) invariant metric on a Lie group induces, by a reduction process, a canonical flow on the *dual of its Lie algebra*. The corresponding evolution equation, known as the *Euler equation*, is Hamiltonian relatively to some canonical *Poisson structure*. It generalizes the Euler equation of the free motion of a rigid body[1]. In a famous article [1], Arnold pointed out that this formalism could be applied to the group of volume-preserving diffeomorphisms to describe the motion of an

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[1]In that case, the group is just the rotation group, $SO(3)$.

ideal fluid[2]. Thereafter, it became clear that many equations from mathematical physics could be interpreted the same way.

In [19] (see also [44]), Dorfman and Gelfand showed that Korteweg-de Vries [27] equation can be obtained as the geodesic equation, on the Virasoro group, of the right-invariant metric defined on the Lie algebra by the L^2 inner product. In [41], Misiolek has shown that Camassa-Holm equation [4] which is also a one dimensional model for shallow water waves, can be obtained as the geodesic flow on the Virasoro group for the H^1-metric.

While both the Korteweg-de Vries and the Camassa-Holm equation have a geometric derivation and both are models for the propagation of shallow water waves, the two equations have quite different structural properties. For example, while all smooth periodic initial data for the Korteweg-de Vries equation develop into periodic waves that exist for all times [48], smooth periodic initial data for the Camassa-Holm equation develop either into global solutions or into breaking waves (see the papers [5, 8, 9, 39]).

In this paper, we study the case of right-invariant metrics on the diffeomorphism group of the circle, $\mathrm{Diff}(S^1)$. Notice however that a similar theory is likely without the periodicity condition (in which case, some weighted spaces express how close the diffeomorphisms of the line are to the identity [7]).

Each right-invariant metric on $\mathrm{Diff}(S^1)$ is defined by an inner product $\mathbf{a}$ on the Lie algebra of the group, $\mathrm{Vect}(S^1) = C^\infty(S^1)$. If this inner product is *local*, it is given by the expression

$$\mathbf{a}(u, v) = \int_{S^1} u\, A(v)\, dx \qquad u, v \in C^\infty(S^1),$$

where A is an invertible, symmetric, linear differential operator. To this inner product on $\mathrm{Vect}(S^1)$, corresponds a quadratic functional (the energy functional)

$$H_A(m) = \frac{1}{2} \int_{S^1} m\, A^{-1}(m),$$

on the (regular) dual $\mathrm{Vect}^*(S^1)$. Its corresponding Hamiltonian vector field X_A generates the Euler equation

$$\frac{dm}{dt} = X_A(m).$$

Among Euler equations of that kind, we have the well-known *inviscid Burgers* equation

$$u_t + 3uu_x = 0,$$

and *Camassa-Holm* [4, 16] shallow water equation

$$u_t + uu_x + \partial_x\, (1 - \partial_x^2)^{-1} \Big(u^2 + \frac{1}{2}\, u_x^2 \Big) = 0.$$

Indeed, the inviscid Burgers equation corresponds to $A = I$ (L^2 inner product), whereas the Camassa-Holm equation corresponds to $A = I - D^2$ (H^1 inner product) (see [10, 11]).

Burgers, Korteweg-de Vries and of Camassa-Holm equations are precisely bi-Hamiltonian relatively to some second *affine* (after Souriau [47]) compatible Poisson structure[3] (see

[2]However, this formalism seems to have been extended to hydrodynamics before Arnold by Moreau [42].

[3]The affine structure on the Virasoro algebra which makes Korteweg-de Vries equation a bi-Hamiltonian system seems to have been first discovered by Gardner [17] and for this reason, some authors call it the *Gardner bracket* (see also [15].

[14, 32, 37]). Since these equations are special cases of Euler equations induced by H^k-metric, it is natural to ask whether, in general, these equations have similar properties for any value of k. In [12], it was shown that this *was not the case*. There are no affine structure on $\mathrm{Vect}^*(S^1)$ which makes the Eulerian vector field X_k, generated by the H^k-metric, a bi-Hamiltonian system, unless $k = 0$ (Burgers) or $k = 1$ (Camassa-Holm). One similar result for the Virasoro algebra was given in [13]. We investigate, here, the problem of finding a modified Lie-Poisson structure for which the vector field X_A is bi-Hamiltonian. We show, in particular, that for an operator A with constant coefficients, this is possible only if $A = aI + bD^2$, where $a, b \in \mathbb{R}$.

In §2, we recall the definition of Hamiltonian and bi-Hamiltonian manifolds and the basic materials on bi-Hamiltonian vector fields. Section 3 contains a description of Poisson structures on the dual of the Lie algebra of a Lie group. The last section is devoted to the study of bi-Hamiltonian Euler equations on $\mathrm{Vect}^*(S^1)$; the main results are stated and proved.

In the description of modified affine Poisson structures we rely on Gelfand-Fuks cohomology. Since the handling of this cohomology theory is not obvious, we derive, in the Appendix, an elementary, "hands-on" computation of the two first Gelfand-Fuks cohomological groups of $\mathrm{Vect}(S^1)$.

2. Hamiltonian and bi-Hamiltonian manifolds

In this section, we recall definitions and well-known results on finite dimensional smooth Poisson manifolds.

2.1. Poisson manifolds.

Definition 2.1. A *symplectic manifold* is a pair (M, ω), where M is a manifold and ω is a closed nondegenerate 2-form on M, that is $\mathrm{d}\omega = 0$ and for each $m \in M$, ω_m is a non degenerate bilinear skew-symmetric map of $T_m M$.

Since a symplectic form ω is nondegenerate, it induces an isomorphism

$$TM \to T^*M, \quad X \mapsto i_X\, \omega, \tag{1}$$

defined via $i_X\, \omega(Y) = \omega(X, Y)$. For example, this allows to define the *symplectic gradient* X_f of a function f by the relation $i_{X_f}\, \omega = -\mathrm{d}f$. The inverse of this isomorphism (1) defines a skew-symmetric bilinear form P on the cotangent space T^*M. This bilinear form P induces itself a bilinear mapping on $C^\infty(M)$, the space of smooth functions $f : M \to \mathbb{R}$, given by

$$\{f, g\} = P(\mathrm{d}f, \mathrm{d}g) = \omega(X_f, X_g), \quad f, g \in C^\infty(M), \tag{2}$$

and called the *Poisson bracket* of the functions f and g.

The observation that a bracket like (2) could be introduced on $C^\infty(M)$ for a smooth manifold M, without the use of a symplectic form, leads to the general notion of a *Poisson structure* [34].

Definition 2.2. A *Poisson (or Hamiltonian*[4]*) structure* on a C^∞ manifold M is a skew-symmetric bilinear mapping $(f, g) \mapsto \{f, g\}$ on the space $C^\infty(M)$, which satisfies the

[4]The expression *Hamiltonian manifold* is often used for the generalization of Poisson structure in the case of infinite dimension manifolds.

Jacobi identity

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0, \tag{3}$$

as well as the *Leibnitz identity*

$$\{f,gh\}=\{f,g\}h+g\{f,h\}. \tag{4}$$

When the Poisson structure is induced by a symplectic structure ω, the *Leibnitz identity* is a direct consequence of (2), whereas the *Jacobi identity* (3) corresponds to the condition $\mathrm{d}\omega=0$ satisfied by the symplectic form ω. In the general case, the fact that the mapping $g\mapsto\{f,g\}$ satisfies (4) means that it is a *derivation* of $C^\infty(M)$.

Each derivation on $C^\infty(M)$ corresponds to a smooth vector field, that is, to each $f\in C^\infty(M)$ is associated a vector field $X_f: M\to TM$, called the *Hamiltonian vector field* of f, such that

$$\{f,g\}=X_f\cdot g=L_{X_f}\,g, \tag{5}$$

where $L_{X_f}\,g$ is the *Lie derivative* of g along X_f.

Jost [24] pointed out that, just like a derivation on $C^\infty(M)$ corresponds to a vector field, a bilinear bracket $\{f,g\}$ satisfying the Leibnitz rule (4) corresponds to a field of bivectors. That is, there exists a C^∞ tensor field $P\in\Gamma(\bigwedge^2 TM)$, called the *Poisson bivector* of $(M,\{\cdot,\cdot\})$, such that

$$\{f,g\}=P(\mathrm{d}f,\mathrm{d}g). \tag{6}$$

for all $f,g\in C^\infty(M)$.

Proposition 2.3. *A bivector field $P\in\Gamma(\bigwedge^2 TM)$ is the* Poisson bivector *of a Poisson structure on M if and only if one of the following equivalent conditions holds:*

1. $[P,P]=0$, *where* $[\ ,\ ]$ *is the* Schouten-Nijenhuis bracket[5]*,*
2. *The bracket $\{f,g\}=P(\mathrm{d}f,\mathrm{d}g)$ satisfies the Jacobi identity,*
3. $[X_f,X_g]=X_{\{f,g\}}$*, for all $f,g\in C^\infty(M)$.*

Proof. By definition of the Schouten-Nijenhuis bracket [49], we have

$$\begin{aligned}
-\frac{1}{2}[P,P](\mathrm{d}f,\mathrm{d}g,\mathrm{d}h) &= \circlearrowleft P(\mathrm{d}Q(\mathrm{d}f,\mathrm{d}g),\mathrm{d}h)\\
&=\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}\\
&=X_{\{f,g\}}\cdot h - X_f\cdot X_g\cdot h + X_g\cdot X_f\cdot h
\end{aligned}$$

for all $f,g,h\in C^\infty(M)$ where $\circlearrowleft$ indicates the sum over circular permutations of f,g,h. Hence, all these expressions vanish together. □

Remark 2.4. The notion of a Poisson manifold is more general than that of a symplectic manifold. Symplectic structures correspond to nondegenerate Poisson structure. In that case, the Poisson bracket satisfies the additional property that $\{f,g\}=0$ for all $g\in C^\infty(M)$ only if $f\in C^\infty(M)$ is a constant, whereas for Poisson manifolds such non-constant functions f might exist, in which case they are called *Casimir functions*. Such functions are constants of motion for all vector fields X_g where $g\in C^\infty(M)$.

[5]The Schouten-Nijenhuis bracket is an extension of the Lie bracket of vector fields to skew-symmetric multivector fields, see [49].

On a Poisson manifold (M, P), a vector field $X : M \to TM$ is said to be *Hamiltonian* if there exists a function f such that $X = X_f$. On a symplectic manifold (M, ω), a necessary condition for a vector field X to be Hamiltonian is that

$$L_X \omega = 0.$$

A similar criterion exists for a Poisson manifold (M, P) (see [49]). A necessary condition for a vector field X to be *Hamiltonian* is

$$L_X P = 0.$$

2.2. **Integrability.** An *integrable system* on a symplectic manifold M of dimension $2n$ is a set of n functionally independent[6] $f_1, \ldots, f_n$ which are *in involution*, i.e. such that

$$\forall j, k \qquad \{f_j, f_k\} = 0.$$

A Hamiltonian vector field X_H is said to be *(completely) integrable* if the Hamiltonian function H belongs to an integrable system. In other words, X_H is integrable if there exists n first integrals[7] of X_H, $f_1 = H, f_2, \ldots, f_n$ which commute together.

Remark 2.5. At any point x where the functions $f_1, \ldots, f_n$ are functionally independent, the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ generate a *maximal isotropic* subspace L_x of $T_x M$. When x varies, the subspaces generate what one calls a *Lagrangian distribution*; that is a sub-bundle L of TM whose fibers are maximal isotropic subspaces. In our case, this distribution is integrable (in the sense of Frobenius). The leaves of L are defined by the equations

$$f_1 = \text{const.}, \ldots, f_n = \text{const.}.$$

A Lagrangian distribution which is integrable (in the sense of Frobenius) is called a *real polarization* and is a key notion in *Geometric Quantization*.

In the study of dynamical systems, the importance of integrable Hamiltonian vector fields is emphasized by the *Arnold-Liouville theorem* [2] which asserts that each compact leaf is actually diffeomorphic to an n-dimensional torus

$$T^n = \left\{(\varphi^1, \ldots, \varphi^n); \quad \varphi^k \in \mathbb{R}/2\pi\mathbb{Z}\right\},$$

on which the flow of X_H defines a linear quasi-periodic motion, i.e. that in angular coordinates $(\varphi^1, \ldots, \varphi^n)$

$$\frac{d\varphi^k}{dt} = \omega^k, \quad k = 0, \ldots, n,$$

where $(\omega^1, \ldots, \omega^n)$ is a constant vector.

Remark 2.6. In the case of a Poisson manifold, it can be confusing to define an integrable system. However, we can use the symplectic definition on each symplectic leaves of the Poisson manifold.

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[6]This means that the corresponding Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are independent on an open dense subset of M.

[7]A first integral is a function which is constant on the trajectories of the vector field.

2.3. **Bi-Hamiltonian manifolds.** Two Poisson brackets $\{\ ,\ \}_P$ and $\{\ ,\ \}_Q$ are *compatible* if any linear combination

$$\{f,g\}_{\lambda,\mu} = \lambda\{f,g\}_P + \mu\{f,g\}_Q, \qquad \lambda,\mu \in \mathbb{R},$$

is also a Poisson bracket. A *bi-Hamiltonian manifold* (M,P,Q) is a manifold equipped with two Poisson structures P and Q which are compatible.

Proposition 2.7. *Let P and Q be two Poisson structures on M. Then P and Q are compatible if and only if one of the following equivalent conditions holds:*

(1) $[P,Q]=0$*, where* $[\ ,\ ]$ *is the Schouten-Nijenhuis bracket,*
(2) $\circlearrowleft \{\{g,h\}_P, f\}_Q + \{\{g,h\}_Q, f\}_P = 0$*, where* $\circlearrowleft$ *is the sum over circular permutations of* f,g,h*,*
(3) $[X_f^P, X_g^Q] + [X_f^Q, X_g^P] = X^P_{\{f,g\}_Q} + X^Q_{\{f,g\}_P}$*, for all* $f,g \in C^\infty(M)$*.*

Proof. By definition of the Schouten-Nijenhuis bracket [49], we have

$$\begin{aligned}
-[P,Q](\mathrm{d}f,\mathrm{d}g,\mathrm{d}h) &= \circlearrowleft P(\mathrm{d}Q(\mathrm{d}f,\mathrm{d}g),\mathrm{d}h) + Q(\mathrm{d}P(\mathrm{d}f,\mathrm{d}g),\mathrm{d}h)\\
&= \circlearrowleft \{\{g,h\}_P, f\}_Q + \{\{g,h\}_Q, f\}_P\\
&= -[X_f^P, X_g^Q]\cdot h - [X_f^Q, X_g^P]\cdot h\\
&\quad + X^P_{\{f,g\}_Q}\cdot h + X^Q_{\{f,g\}_P}\cdot h
\end{aligned}$$

for all $f,g,h \in C^\infty(M)$. Hence, all these expressions vanish together. □

2.4. **Lenard recursion relations.** On a bi-Hamiltonian manifold M, equipped with two compatible Poisson structures P and Q, we say that a vector field X is (formally) *integrable*[8] or *bi-Hamiltonian* if it is Hamiltonian for both structures. The reason for this terminology is that for such a vector field, there exists under certain conditions a hierarchy of first integrals in involution that may lead in certain case to complete integrability, in the sense of Liouville. A useful concept for obtaining such a hierarchy of first integrals is the so called *Lenard scheme* [38].

Definition 2.8. On a manifold M equipped with two Poisson structures P and Q, we say that a sequence $(H_k)_{k\in\mathbb{N}^*}$ of smooth functions satisfy the *Lenard recursion relation* if

$$P\,dH_k = Q\,dH_{k+1}, \tag{7}$$

for all $k \in \mathbb{N}^*$.

Proposition 2.9. *Let P and Q be Poisson structures on a manifold M and let $(H_k)_{k\in\mathbb{N}^*}$ be a sequence of smooth functions on M that satisfy the Lenard recursion relation. Then the functions, H_k, are pairwise in involution with respect to both brackets P and Q.*

Proof. Using skew-symmetry of P and Q and relation (7), we get

$$P(dH_k, dH_{k+p}) = Q(dH_{k+1}, dH_{k+p}) = P(dH_{k+1}, dH_{k+p-1}),$$

for all $k,p \in \mathbb{N}^*$. From which we deduce, by induction on p, that

$$\{H_k, H_{k+p}\}_P = 0,$$

for all $k,p \in \mathbb{N}^*$. It is then an immediate consequence that

$$\{H_k, H_l\}_Q = 0,$$

[8] This terminology is used for evolution equations in infinite dimension.

for all $k, l \in \mathbb{N}^*$. □

Remark 2.10. Notice that in the proof of proposition 2.9, the compatibility of P and Q is not needed.

Suppose now that (M, P, Q) is a bi-Hamiltonian manifold and that at least one of the two Poisson brackets, say Q is *invertible*. In that case, we can define a $(1,1)$-tensor field

$$R = PQ^{-1},$$

which is called the *recursion operator* of the bi-Hamiltonian structure. It has been shown [28, 29] that, as a consequence of the compatibility of P and Q, the *Nijenhuis torsion* of R, defined by

$$T(R)(X,Y) = [RX, RY] - R\big([RX, Y] + [X, RY]\big) + R^2[X,Y]$$

vanishes. In this situation, the family of Hamiltonians

$$H_k = \frac{1}{k}\operatorname{tr} R^k, \qquad (k \in \mathbb{N}^*),$$

satisfy the Lenard recursion relation (7). Indeed, this results from the fact that

$$L_X \operatorname{tr}(T) = \operatorname{tr}(L_X T)$$

for every vector field X and every $(1,1)$-tensor field T on M and that the vanishing of the Nijenhuis torsion of R can be rewritten as

$$L_{RX} R = R\, L_X\, R$$

for all vector field X.

Remark 2.11. This construction has to be compared with *Lax isospectral equation* associated to an evolution equation

$$\frac{du}{dt} = F(u). \tag{8}$$

The idea is to associate to equation (8), a pair of matrices (or operators in the infinite dimensional case) (L, B), called a *Lax pair*, whose coefficients are functions of u and in such a way that when $u(t)$ varies according to (8), $L(t) = L(u(t))$ varies according to

$$\frac{dL}{dt} = [L, B].$$

This equation has been formulated in [30] in order to obtain a hierarchy of first integrals of the evolution equation as eigenvalues or traces of the operator L. This analogy between R and L is not casual and has been studied in [29]. Many evolution equations which admit a Lax pair appear to be also bi-Hamiltonian systems generated by a recursion operator $R = PQ^{-1}$.

In practice, we may be confronted to the following problem. We start with an evolution equation represented by a vector field X on a manifold M. We find two compatible Poisson structures P and Q on M which makes X a bi-Hamiltonian vector field. But P and Q are *both non-invertible*. In that case, it is however still possible to find a Lenard hierarchy if the following algorithm works.

Step 1: Let H_1 the Hamiltonian of X for the Poisson structure P and let $X_1 = X$. The vector field X_1 is Hamiltonian for the Poisson structure Q by assumption, this defines

Hamiltonian function H_2. We define X_2 to be the Hamiltonian vector field generated by H_2 for the Poisson structure P.

Step 2: Inductively, having defined Hamiltonian function H_k and letting X_k be the Hamiltonian vector field generated by H_k for the Poisson structure P, we check if X_k is Hamiltonian for the Poisson structure Q. If the answer is yes, then we define H_{k+1} to be the Hamiltonian of X_k for the Poisson structure Q.

3. Poisson structures on the dual of a Lie algebra

3.1. **Lie-Poisson structure.** The fundamental example of a non-symplectic Poisson structure is the *Lie-Poisson structure* on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$.

Definition 3.1. On the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ of a Lie group G, there is a Poisson structure defined by

$$\{f, g\}(m) = m([\mathrm{d}_m f, \mathrm{d}_m g]) \tag{9}$$

for $m \in \mathfrak{g}^*$ and $f, g \in C^\infty(\mathfrak{g}^*)$, called the *canonical Lie-Poisson structure*[9].

Remark 3.2. The canonical Lie-Poisson structure has the remarkable property to be *linear*, that is the bracket of two linear functionals is itself a linear functional. Given a basis of $\mathfrak{g}$, the components[10] of the Poisson bivector W associated to (9) are

$$P_{ij} = c_{ij}^k \, x_k, \tag{10}$$

where c_{ij}^k are the *structure component* of the Lie algebra $\mathfrak{g}$.

3.2. **Modified Lie-Poisson structures.** Under the general name of *modified Lie-Poisson structures*, we mean an affine[11] perturbation of the canonical Lie-Poisson structure on $\mathfrak{g}^*$. In other words, it is represented by a bivector

$$P + Q,$$

where P is the canonical Poisson bivector defined by (10) and $Q = (Q_{ij})$ is a constant bivector on $\mathfrak{g}^*$. Such a $Q \in \bigwedge^2 \mathfrak{g}^*$ is itself a Poisson bivector. Indeed the Schouten-Nijenhuis bracket

$$[Q, Q] = 0,$$

since Q is a constant tensor field on $\mathfrak{g}^*$.

The fact that $P + Q$ is a Poisson bivector, or equivalently that Q is compatible with the canonical Lie-Poisson structure, is expressed using proposition 2.7, by the condition

$$Q([u, v], w) + Q([v, w], u) + Q([w, u], v) = 0, \tag{11}$$

for all $u, v, w \in \mathfrak{g}$.

[9]Here, $\mathrm{d}_m f$, the differential of a function $f \in C^\infty(\mathfrak{g}^*)$ at $m \in \mathfrak{g}^*$ is to be understood as an element of the Lie algebra $\mathfrak{g}$

[10]In what follows, the convention for lower or upper indices may be confusing since we shall deal with tensors on both $\mathfrak{g}$ and $\mathfrak{g}^*$. Therefore, we emphasize that the convention we use in this paper is the following: upper-indices correspond to contravariant tensors on $\mathfrak{g}$ and therefore covariant tensors on $\mathfrak{g}^*$ whereas lower indices correspond to covariant tensors on $\mathfrak{g}$ and therefore contravariant tensors on $\mathfrak{g}^*$.

[11]A Poisson structure on a linear space is *affine* if the bracket of two linear functionals is an affine functional.

3.3. **Lie algebra cohomology.** On a Lie group G, a left-invariant[12] p-form ω is completely defined by its value at the unit element e, and hence by an element of $\bigwedge^p \mathfrak{g}^*$. In other words, there is a natural isomorphism between the space of left-invariant p-forms on G and $\bigwedge^p \mathfrak{g}^*$. Moreover, since the exterior differential d commutes with left translations, it induces a linear operator $\partial : \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+1} \mathfrak{g}^*$ defined by

$$\partial\gamma(u_0, \dots, u_p) = \sum_{i<j} (-1)^{i+j} \gamma([u_i, u_j], u_0, \dots, \widehat{u_i}, \dots, \widehat{u_j}, \dots, u_p), \tag{12}$$

where the hat means that the corresponding element should not appear in the list. γ is said to be a *cocycle* if $\partial\gamma = 0$. It is a *coboundary* if is of the form $\gamma = \partial\mu$ for some cochain μ in dimension $p-1$. Every coboundary is a cocycle: that is $\partial \circ \partial = 0$.

Example 3.3. For every $\gamma \in \bigwedge^0 \mathfrak{g}^* = \mathbb{R}$, we have $\partial\gamma = 0$. For $\gamma \in \bigwedge^1 \mathfrak{g}^* = \mathfrak{g}^*$, we have

$$\partial\gamma\,(u, v) = -\gamma([u, v]),$$

where $u, v \in \mathfrak{g}$. For $\gamma \in \bigwedge^2 \mathfrak{g}^*$, we have

$$\partial\gamma\,(u, v, w) = -\gamma([u, v], w) - \gamma([v, w], u) - \gamma([w, u], v),$$

where $u, v, w \in \mathfrak{g}$.

The kernel $Z^p(\mathfrak{g})$ of $\partial : \bigwedge^p(\mathfrak{g}^*) \to \bigwedge^{p+1}(\mathfrak{g}^*)$ is the space of *p-cocycles* and the range $B^p(\mathfrak{g})$ of $\partial : \bigwedge^{p-1}(\mathfrak{g}^*) \to \bigwedge^p(\mathfrak{g}^*)$ is the spaces of *p-coboundaries*. The quotient space $H^p_{CE}(\mathfrak{g}) = Z^p(\mathfrak{g})/B^p(\mathfrak{g})$ is the p-th *Lie algebra cohomology* or *Chevaley-Eilenberg cohomology group* of $\mathfrak{g}$. Notice that in general the Lie algebra cohomology is different from the de Rham cohomology H^p_{DR}. For example, $H^1_{DR}(\mathbb{R}) = \mathbb{R}$ but $H^1_{CE}(\mathbb{R}) = 0$.

Remark 3.4. Each 2-cocycle γ defines a modified Lie-Poisson structure on $\mathfrak{g}^*$. The compatibility condition (11) can be recast as $\partial\gamma = 0$. Notice that the Hamiltonian vector field X_f of a function $f \in C^\infty(\mathfrak{g}^*)$ computed with respect to the Poisson structure defined by the 2-cocycle γ is

$$X_f(m) = \gamma(\mathrm{d}_m f, \cdot). \tag{13}$$

Example 3.5. A special case of modified Lie-Poisson structure is given by a 2-cocycle γ which is a coboundary. If $\gamma = \partial m_0$ for some $m_0 \in \mathfrak{g}^*$, the expression

$$\{f, g\}_0(m) = m_0([\mathrm{d}_m f, \mathrm{d}_m g])$$

looks like if the Lie-Poisson bracket had been "frozen" at a point $m_0 \in \mathfrak{g}^*$ and for this reason some authors call it a *freezing* structure.

4. Bi-Hamiltonian vector fields on $\mathrm{Vect}^*(S^1)$

4.1. **The Lie algebra** $\mathrm{Vect}(S^1)$**.** The group $\mathfrak{D}$ of smooth orientation-preserving diffeomorphisms of the circle $\mathbb{S}^1$ is endowed with a smooth manifold structure based on the *Fréchet space* $C^\infty(S^1)$. The composition and the inverse are both smooth maps $\mathfrak{D} \times \mathfrak{D} \to \mathfrak{D}$, respectively $\mathfrak{D} \to \mathfrak{D}$, so that $\mathfrak{D}$ is a Lie group [40]. Its Lie algebra $\mathfrak{g}$ is

[12] In this section, we deal with left-invariant forms but, of course, everything we say may be applied equally to right-invariant forms up to a sign in the definition of the coboundary operator.

the space $\mathrm{Vect}(S^1)$ of smooth vector fields on S^1, which is itself isomorphic to the space $C^\infty(S^1)$ of periodic functions. The Lie bracket[13] on $\mathfrak{g} = \mathrm{Vect}(S^1)$ is given by

$$[u, v] = uv_x - u_x v.$$

Lemma 4.1. *The Lie algebra* $\mathrm{Vect}(S^1)$ *is equal to its commutator algebra. That is*

$$\big[\mathrm{Vect}(S^1), \mathrm{Vect}(S^1)\big] = \mathrm{Vect}(S^1).$$

Proof. Any real periodic function u on can be written uniquely as the sum

$$u = w + c$$

where w is periodic function of total integral zero and c is a constant. To be of total integral zero is the necessary and sufficient condition for a periodic function w to have a periodic primitive W. Hence we have $[1, W] = w$. Moreover, since $[\cos, \sin] = 1$, we have proved that every periodic function u can be written as the sum of two commutators. ☐

4.2. **The regular dual** $\mathrm{Vect}^*(S^1)$**.** Since the topological dual of the Fréchet space $\mathrm{Vect}(S^1)$ is too big and not tractable for our purpose, being isomorphic to the space of distributions on the circle, we restrict our attention in the following to the *regular dual* $\mathfrak{g}^*$, the subspace of $\mathrm{Vect}(S^1)^*$ defined by linear functionals of the form

$$u \mapsto \int_{S^1} mu \, dx$$

for some function $m \in C^\infty(S^1)$. The regular dual $\mathfrak{g}^*$ is therefore isomorphic to $C^\infty(S^1)$ by means of the L^2 inner product[14]

$$< u, v >= \int_{S^1} uv \, dx.$$

With these definitions, the *coadjoint action*[15] of the Lie algebra $\mathrm{Vect}(S^1)$ on the regular dual $\mathrm{Vect}^*(S^1)$ is given by

$$ad_u^* m = mu_x + (mu)_x = 2mu_x + m_x u.$$

Let F be a smooth real valued function on $C^\infty(S^1)$. Its *Fréchet* derivative $\mathrm{d}F(m)$ is a linear functional on $C^\infty(S^1)$. We say that F is a *regular function* if there exists a smooth map $\delta F : C^\infty(S^1) \to C^\infty(S^1)$ such that

$$\mathrm{d}F(m)\, M = \int_{S^1} M \cdot \delta F(m) \, dx, \qquad m, M \in C^\infty(S^1).$$

That is, the Fréchet derivative $\mathrm{d}F(m)$ belongs to the regular dual $\mathfrak{g}^*$ and the mapping $m \mapsto \delta F(m)$ is smooth. The map δF is a vector field on $C^\infty(S^1)$, called the *gradient* of F for the L^2-metric. In other words, a regular function is a smooth function on $C^\infty(S^1)$ which has a smooth L^2 gradient.

[13] It corresponds to the Lie bracket of right-invariant vector fields on the group.

[14] In the sequel, we use the notation $u, v, \dots$ for elements of $\mathfrak{g}$ and $m, n, \dots$ for elements of $\mathfrak{g}^*$ to distinguish them, although they all belong to $C^\infty(S^1)$.

[15] The coadjoint action of a Lie algebra $\mathfrak{g}$ on its dual is defined as

$$(ad_u \, m, v) = -(m, ad_u \, v) = -(m, [u, v]),$$

where $u, v \in \mathfrak{g}$, $m \in \mathfrak{g}^*$ and the pairing is the standard one between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Example 4.2. Typical examples of *regular functions* on the space $C^\infty(S^1)$ are *linear functionals*

$$F(m) = \int_{S^1} um\,dx,$$

where $u \in C^\infty(S^1)$. In that case, $\delta F(m) = u$. Other examples are *nonlinear polynomial functionals*

$$F(m) = \int_{S^1} Q(m)\,dx,$$

where Q is a polynomial in derivatives of m up to a certain order r. In that case,

$$\delta F(m) = \sum_{k=0}^{r} (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial Q}{\partial X_k}(m)\right).$$

Notice that the smooth function $F_\theta : C^\infty(S^1) \to \mathbb{R}$ defined by $F_\theta(m) = m(\theta)$ for some fixed $\theta \in \mathbb{S}^1$ is not regular since $\mathrm{d}F_\theta$ is the Dirac measure at θ.

A smooth vector field X on $\mathfrak{g}^*$ is called a *gradient* if there exists a *regular function* F on $\mathfrak{g}^*$ such that $X(m) = \delta F(m)$ for all $m \in \mathfrak{g}^*$. Observe that if F is a smooth real valued function on $C^\infty(S^1)$ then its second Fréchet derivative is symmetric [23], that is,

$$\mathrm{d}^2F(m)(M,N) = \mathrm{d}^2F(m)(N,M), \qquad m, M, N \in C^\infty(S^1).$$

For a regular function, this property can be rewritten as

$$\int_{S^1} \Big(\delta F'(m)M\Big) N\,dx = \int_{S^1} \Big(\delta F'(m)N\Big) M\,dx, \tag{14}$$

for all $m, M, N \in C^\infty(S^1)$. That is, the linear operator $\delta F'(m)$ is symmetric for the L^2-inner product on $C^\infty(S^1)$ for each $m \in C^\infty(S^1)$. Conversely, a smooth vector field X on $\mathfrak{g}^*$ whose Fréchet derivative $X'(m)$ is a symmetric linear operator is the gradient of the function

$$F(m) = \int_0^1 < X(tm), m > \,dt. \tag{15}$$

This can be checked directly, using the symmetry of $X'(m)$ and an integration by part. We will resume this fact in the following lemma.

Lemma 4.3. *On the Fréchet space $C^\infty(S^1)$ equipped with the (weak) L^2 inner product, a necessary and sufficient condition for a smooth vector field X to be a* gradient *is that its Fréchet derivative $X'(m)$ is a symmetric linear operator.*

4.3. **Hamiltonian structures on** $\mathrm{Vect}^*(S^1)$**.** To define a *Poisson bracket* on the space of *regular functions* on $\mathfrak{g}^*$, we consider a one-parameter family of linear operators P_m ($m \in C^\infty(S^1)$) and set

$$\{F, G\}(m) = \int_{S^1} \delta F(m)\, P_m\, \delta G(m)\,dx. \tag{16}$$

The operators P_m must satisfy certain conditions in order for (16) to be a valid Poisson structure on the regular dual $\mathfrak{g}^*$.

Definition 4.4. A family of linear operators P_m on $\mathfrak{g}^*$ define a Poisson structure on $\mathfrak{g}^*$ if (16) satisfies

(1) $\{F, G\}$ is regular if F and G are regular,
(2) $\{G, F\} = -\{F, G\}$,
(3) $\{\{F, G\}, h\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$.

Notice that the second condition above simply means that P_m is a skew-symmetric operator for each m.

Example 4.5. The canonical Lie-Poisson structure on $\mathfrak{g}^*$ given by

$$\{F, G\}(m) = m\left(\langle \delta F, \delta G\rangle\right) = \int_{S^1} \delta F(m)\,(mD + Dm)\,\delta G(m)\,dx$$

is represented by the one-parameter family of skew-symmetric operators

$$P_m = mD + Dm \tag{17}$$

where $D = \partial_x$. It can be checked that all the three required properties are satisfied. In particular, we have

$$\delta\{F, G\} = \delta F'(P_m \delta G) - \delta G'(P_m \delta F) + \delta F\, D\delta G - \delta G\, D\delta F.$$

Definition 4.6. The *Hamiltonian* of a *regular* function F, for a Poisson structure defined by P is defined as the vector field

$$X_F(m) = P\,\delta F(m).$$

Proposition 4.7. *A necessary condition for a smooth vector field X on $\mathfrak{g}^*$ to be* Hamiltonian *with respect to the Poisson structure defined by a* constant *linear operator Q is the symmetry of the operator $X'(m)Q$ for each $m \in \mathfrak{g}^*$.*

Proof. If X is Hamiltonian, we can find a regular function F such that

$$X(m) = Q\delta F(m).$$

Moreover, since Q is a constant linear operator, we have

$$X'(m) = Q\delta F'(m),$$

and therefore, we get

$$X'(m)Q = Q\delta F'(m)Q,$$

which is a symmetric operator since Q is skew-symmetric and $\delta F'(m)$ is symmetric. □

4.4. **Hamiltonian vector fields generated by right-invariant metrics.** A right-invariant metric on the diffeomorphism group $Diff(S^1)$ is uniquely defined by its restriction to the tangent space to the group at the unity, hence by a *non-degenerate continuous inner product* $\mathbf{a}$ on $\mathrm{Vect}(S^1)$. If this inner product $\mathbf{a}$ is *local*, then according to Peetre [45], there exists a linear differential operator

$$A = \sum_{j=0}^{N} a_j \frac{d^j}{dx^j} \tag{18}$$

where $a_j \in C^\infty(S^1)$ for $j = 0, \dots, N$, such that

$$\mathbf{a}(u, v) = \int_{S^1} A(u)\,v\,dx = \int_{S^1} A(v)\,u\,dx,$$

for all $u, v \in \mathrm{Vect}(S^1)$. The condition for $\mathbf{a}$ to be non-degenerate is equivalent for A to be a *continuous linear isomorphism* of $C^\infty(S^1)$.

Remark 4.8. In the special case where A has *constant coefficients*, the *symmetry* is traduced by the fact that A contains only even derivatives and the *non-degeneracy* by the fact that the *symbol* of A

$$s_A(\xi) = \mathrm{e}^{\mathrm{i}x\xi} A(\mathrm{e}^{-\mathrm{i}x\xi}) = \sum_{j=0}^{N} a_{2j}(-\mathrm{i}\xi)^{2j},$$

has no root in $\mathbb{Z}$.

The right-invariant metric on $\mathrm{Diff}(S^1)$ induced by a continuous, linear, invertible operator A gives rise to an *Euler equation*[16] on $\mathrm{Vect}(S^1)^*$

$$\frac{dm}{dt} = 2mu_x + m_x u, \tag{19}$$

where $m = Au$. This equation is Hamiltonian with respect to the Lie-Poisson structure on $\mathrm{Vect}(S^1)^*$ with Hamiltonian function on $\mathrm{Vect}(S^1)^*$ given by

$$H_2(m) = \frac{1}{2}\int_{S^1} mu\, dx.$$

The corresponding Hamiltonian vector field X_A is given by

$$X_A(m) = (mD + Dm)(A^{-1}m) = 2mu_x + um_x.$$

Remark 4.9. The family of operators

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}},$$

corresponding respectively to the Sobolev H^k inner product, have been studied in [10, 11]. The *Riemannian exponential map* of the corresponding geodesic flow has been shown to be a local diffeomorphism except for $k = 0$. This later case corresponds to the L^2 metric on $\mathrm{Diff}(S^1)$ and happens to be *singular*.

Remark 4.10. A non-invertible inertia operator A may induce in some cases, a weak Riemannian metric on a *homogenous space*. This is the way to interpret Hunter-Saxton and Harry Dym equations as Euler equations, see [25].

The following theorem is a generalization of [12, Theorem 3.7].

Theorem 4.11. *The only continuous, linear, invertible operators*

$$A : \mathrm{Vect}(S^1) \to \mathrm{Vect}(S^1)^*$$

with constant coefficients*, whose corresponding Euler vector field X_A is bi-Hamiltonian relatively to some modified Lie-Poisson structure are*

$$A = aI + bD^2,$$

[16] The second order geodesic equation corresponding to a one sided invariant metric on a Lie group can always be reduced to a first order quadratic equation on the dual of the Lie algebra of the group: the Euler equation (see [3] or [26]). The generality of this reduction was first revealed by Arnold [1].

where $a, b \in \mathbb{R}$ satisfy $a - bn^2 \neq 0, \forall n \in \mathbb{Z}$. The second Hamiltonian structure is induced by the operator

$$Q = DA = aD + bD^3,$$

where $D = d/dx$ and the Hamiltonian function is

$$H_3(m) = \frac{1}{2}\int_{S^1} \left(au^3 - bu(u_x)^2\right) dx,$$

where $m = Au$.

Remark 4.12. We insist on the fact that the proof we give applies for an operator with *constant coefficients*. It would be interesting to study the case of an invertible, continuous linear operator whose coefficients are *not constant*. Are there such operator A with bi-Hamiltonian Euler vector field X_A relative to some modified Lie-Poisson structure ? In that case, for which modified Lie-Poisson structures Q is there an Euler vector field X_A which is bi-Hamiltonian relatively to Q ?

Proof. The proof is essentially the same as the one given in [12]. A direct computation shows that

$$X_A(m) = (aD + bD^3)\,\delta H_3(m)$$

where

$$H_3(m) = \frac{1}{2}\int_{S^1} \left(au^3 - bu(u_x)^2\right) dx,$$

and

$$A = aI + bD^2,$$

where $a, b \in \mathbb{R}$.

Each modified Lie-Poisson structure on $\mathrm{Vect}^*(S^1)$ is given by a *local* 2-*cocycle* of $\mathrm{Vect}(S^1)$. According to proposition A.3 (see the Appendix), such a cocycle is represented by a differential operator

$$Q = m_0 D + D m_0 + \beta D^3 \tag{20}$$

where $m_0 \in C^\infty(S^1)$ and $\beta \in \mathbb{R}$. We will now show that there is no such cocycle for which X_A is Hamiltonian if the order of

$$A = \sum_{j=0}^{N} a_{2j} D^{2j}$$

is strictly greater than 2.

By virtue of proposition 4.7, a necessary condition for X_A to be Hamiltonian with respect to the cocycle represented by Q is that

$$K(m) = X_A'(m)Q$$

is a symmetric operator. We have

$$X_A'(m) = 2u_x I + uD + 2mDA^{-1} + m_x A^{-1},$$

and in particular, for $m = 1$,

$$X_A'(1) = D + 2DA^{-1}.$$

Hence

$$K(1) = \left(D + 2DA^{-1}\right) \circ \left(m_0 D + D m_0\right) + \beta D^4(1 + 2A^{-1}),$$

whereas

$$K(1)^* = \big(m_0 D + D m_0\big) \circ \big(D + 2DA^{-1}\big) + \beta D^4(1 + 2A^{-1}).$$

Therefore, letting $m_0' = \frac{dm_0}{dx}$, we get

$$K(1) - K(1)^* = \big(m_0' D + D m_0'\big) + 2\big(A^{-1} D m_0 D - D m_0 D A^{-1}\big) + \\ + 2\big(A^{-1} D^2 m_0 - m_0 D^2 A^{-1}\big),$$

and this operator vanishes if and only if

$$A\big(K(1) - K(1)^*\big)A = 0. \tag{21}$$

But $A\big(K(1) - K(1)^*\big)A$ is the sum of 2 linear differential operators:

$$2\big(D m_0 D A - A D m_0 D\big) + 2\big(D^2 m_0 A - A m_0 D^2\big),$$

which is of order less than $2N + 2$ and

$$A\big(m_0' D + D m_0'\big)A,$$

which is of order $4N + 1$ unless $m_0' = 0$ which must be the case if (21) holds. Therefore m_0 has to be a constant. Let $\alpha = 2m_0 \in \mathbb{R}$. Then

$$K(m) = \alpha \left\{2u_x D + u D^2 + 2m D^2 A^{-1} + m_x D A^{-1}\right\} + \\ + \beta \left\{2u_x D^3 + u D^4 + 2m D^4 A^{-1} + m_x D^3 A^{-1}\right\}$$

because D and A commute. The symmetry of the operator $K(m)$ means

$$\int_{S^1} N\, K(m) M\, dx = \int_{S^1} M\, K(m) N\, dx, \tag{22}$$

for all $m, M, N \in C^\infty(S^1)$. Since this last expression is tri-linear in the variables m, M, N, the equality can be checked for complex periodic functions m, M, N. Let $m = Au$, $u = \mathrm{e}^{-\mathrm{i}px}$, $M = \mathrm{e}^{-\mathrm{i}qx}$ and $N = \mathrm{e}^{-\mathrm{i}rx}$ with $p, q, r \in \mathbb{Z}$. We have

$$\int_{S^1} N\, K(m) M\, dx = \Big[(2pq^3 + q^4)\beta - (2pq + q^2)\alpha + \\ + \Big((pq^3 + 2q^4)\beta - (pq + 2q^2)\alpha\Big)\frac{s_A(p)}{s_A(q)}\Big] \int_{S^1} \mathrm{e}^{-\mathrm{i}(p+q+r)x}\, dx\,,$$

whereas

$$\int_{S^1} M\, K(m) N\, dx = \Big[(2pr^3 + r^4)\beta - (2pr + r^2)\alpha + \\ + \Big((pr^3 + 2r^4)\beta - (pr + 2r^2)\alpha\Big)\frac{s_A(p)}{s_A(r)}\Big] \int_{S^1} \mathrm{e}^{-\mathrm{i}(p+q+r)x} dx\,.$$

Now we set $p = n$, $q = -2n$, $r = n$ and we must have

$$(24n^4\beta - 6n^2\alpha)s_A(n) = (6n^4\beta - 6n^2\alpha)s_A(2n), \tag{23}$$

if $K(m)$ is symmetric.

If $\beta \neq 0$, the leading term in the left hand-side of (23) is $24\,(-1)^N\, a_{2N}\, \beta\, n^{2N+4}$, whereas the leading term of the right hand-side is $6\,(-1)^N\, 2^{2N}\, a_{2N}\, \beta\, n^{2N+4}$. Hence, unless $N = 1$, we must have $\beta = 0$.

On the other hand, if $\beta = 0$, we must have $\alpha s_A(n) = \alpha s_A(2n)$, for all $n \in \mathbb{N}^*$. Thus $\alpha = 0$ unless $N = 0$. This completes the proof. $\square$

4.5. **Hierarchy of first integrals.** In view of theorem 4.11, the next step is to find a hierarchy of first integrals in involution for the vector field X_A where

$$A = aI + bD^2,$$

and $a, b \in \mathbb{R}$ satisfy $a - bn^2 \neq 0, \forall n \in \mathbb{Z}$. The vector field

$$X_A(m) = 2mu_x + um_x.$$

is bi-Hamiltonian. It can be written as

$$X_A(m) = P_m \, \delta H_2(m),$$

where

$$H_2(m) = \frac{1}{2} \int_{S^1} um \, dx$$

and $P_m = mD + Dm$ or as

$$X_A(m) = Q \, \delta H_3(m),$$

where

$$H_3(m) = \frac{1}{3} \int_{S^1} u(um + q(u)) \, dx,$$

$q(u) = 1/2(au^2 + bu_x^2)$ and $Q = DA = aD + bD^3$.

The problem we get when we try to apply the Lenard scheme to obtain a hierarchy of conserved integrals is that both Poisson operators P_m and Q are non invertible. However, Q is composed of two commuting operators, A which is invertible and D which is not. The image of D is the codimension 1 subspace, $C_0^\infty(S^1)$, of smooth periodic functions with zero integral. The restriction of D to this subspace is invertible with inverse D^{-1}, the linear operator which associates to a smooth function with zero integral its unique primitive with zero integral. Following Lax in [31], we are able to prove the following result.

Theorem 4.13. *There exists a sequence $(H_k)_{k\in\mathbb{N}^*}$ of functionals, whose gradients G_k are polynomial expressions of $u = A^{-1}m$ and its derivatives, which satisfy the* Lenard recursion scheme

$$P_m \, G_k = Q \, G_{k+1}.$$

Remark 4.14. It is worth to notice, that contrary to the result given by Lax in [31], for the KdV equation, the operators G_k are polynomials in $u = A^{-1}m$ and not in m. In particular, there are non-local operators[17], if $A \neq aI$, for some $a \in \mathbb{R}$.

Before giving a sketch of proof of this theorem, let us illustrate the explicit computation of the first Hamiltonians of the hierarchy. We start with

$$H_1(m) = \int_{S^1} m \, dx, \qquad G_1(m) = 1.$$

We define X_1 to be the Hamiltonian vector field of H_1 for the Lie-Poisson structure P_m

$$X_1(m) = P_m \, G_1(m) = m_x.$$

$X_1(m)$ is in the image of D for all m and we can define

$$G_2(m) = Q^{-1} X_1(m) = A^{-1} D^{-1}(m_x) = A^{-1}(m) = u$$

[17]Notice that our m corresponds to u in the notations of [31].

which is the gradient of the second Hamiltonian of the hierarchy

$$H_2(m) = \frac{1}{2}\int_{S^1} mu\, dx.$$

We compute then X_2, the Hamiltonian vector field of H_2 for P_m

$$X_2(m) = P_m\, G_2(m) = 2mu_x + m_x u = (mu + q(u))_x,$$

where $q(u) = 1/2(au^2 + bu_x^2)$. $X_2(m)$ is in the image of D for all m and we can define

$$G_3(m) = Q^{-1}X_2(m) = A^{-1}(mu + q(u)),$$

which is the gradient of the third Hamiltonian of the hierarchy

$$H_3(m) = \frac{1}{3}\int_{S^1} u(mu + q(u))\, dx.$$

So far, we obtain this way a hierarchy of Hamiltonians $(H_k)_{k\in\mathbb{N}^*}$ satisfying the Lenard recursion relations for the Euler equation associated to the operator A.

Example 4.15 (Burgers Hierarchy). For $A = I$, we obtain explicitly the whole *Burgers hierarchy*

$$H_{k+1}(m) = \frac{(2k!)}{2^k(k!)^2(k+1)}\int_{S^1} m^{k+1}\, dx, \qquad (k \in \mathbb{N}).$$

Example 4.16 (Camassa-Holm Hierarchy). For $A = I - D^2$, we obtain the *Camassa-Holm hierarchy*. The first members of the family are

$$\begin{aligned}
H_1(m) &= \int_{S^1} m\, dx = \int_{S^1} u\, dx,\\
H_2(m) &= \frac{1}{2}\int_{S^1} mu\, dx = \frac{1}{2}\int_{S^1} (u^2 + u_x^2)\, dx,\\
H_3(m) &= \frac{1}{2}\int_{S^1} u(u^2 + u_x^2)\, dx.
\end{aligned}$$

The next integrals of the hierarchy are much harder to compute explicitly. One may consider [33, 35] for further studies on the subject.

Sketch of Proof of Theorem 4.13. The proof is divided into two steps. We refer to [31] for the details.

Step 1: We show by induction that there exists a sequence of vector fields G_k, which are polynomial expressions of $u = A^{-1}m$ and its derivatives and which satisfy

$$G_1 = 1, \qquad PG_k = QG_{k+1}, \quad \forall k \in \mathbb{N}^*. \tag{24}$$

Step 2: We show that G_k is, for all k the gradient of a function H_k.

To prove Step 1, we suppose that $G_1, \dots, G_n$ have been constructed satisfying (24) and we use the following two lemmas[18] to show that G_{n+1} exists.

Lemma 4.17. *Suppose that Q is a polynomial in derivatives of u up to order r such that*

$$\int_{S^1} Q(u)\, dx = 0,$$

for all $u \in C^\infty(S^1)$. Then there exists a polynomial G in derivatives of u up to order $r-1$ such that $Q = DG$.

[18]The proof of lemma 4.17 can be found in [43] while the proof of lemma 4.18 can be found in [31].

Lemma 4.18. *We have*

$$\int_{S^1} PG_n \, dx = 0$$

for all $n \in \mathbb{N}^*$.

To prove Step 2, it is enough to show that G_k' is a symmetric operator for all k, by virtue of Lemma 4.3. We suppose that $G_1, \dots, G_n$ are gradients and show first the following result.

Lemma 4.19. *The operator*

$$QG_{n+1}'(m)Q$$

is symmetric for all $m \in C^\infty(S^1)$.

We conclude then, like in [31], that $G_{n+1}'(m)$ itself is symmetric. We will give here the details of the proof of Lemma 4.19, since the proof of the corresponding result for KdV in [31] is just a direct, hand waving computation and does not apply in our more general case.

Proof of Lemma 4.19. First, we differentiate the recurrence formula (24) and we obtain

$$QG_{n+1}'(m) = ad_{G_n}^* + P_m G_n'(m) \tag{25}$$

and

$$QG_n'(m) = ad_{G_{n-1}}^* + P_m G_{n-1}'(m). \tag{26}$$

We multiply (25) by Q on the right, (26) by P on the right, and subtract (26) from (25); we get

$$QG_{n+1}'(m)Q = QG_n'(m)P_m + P_m G_n'(m)Q + ad_{G_n}^* Q - ad_{G_{n-1}}^* P_m - P_m G_{n-1}'(m) P_m.$$

Using the fact that

$$(ad_u^*)^* = -ad_u,$$

we get finally

$$\big(QG_{n+1}'(m)Q\big)^* - QG_{n+1}'(m)Q = Qad_{G_n} - P_m ad_{G_{n-1}} - ad_{G_n}^* Q + ad_{G_{n-1}}^* P_m.$$

Using the fact that Q satisfy the following cocycle condition

$$Q([u,v]) = ad_u^* Q(v) - ad_v^* Q(u)$$

which can be rewritten as

$$Qad_u = ad_u^* Q - P_{Q(u)},$$

we get

$$\big(QG_{n+1}'(m)Q\big)^* - QG_{n+1}'(m)Q = -P_{Q(G_n)} - P_m ad_{G_{n-1}} + ad_{G_{n-1}}^* P_m.$$

But this last expression is zero because

$$P_m ad_v = ad_v^* P_m - P_{P_m(v)}$$

and $Q(G_n) = P_m G_{n-1}$. □

□

Remark 4.20. In the special case where the cocycle γ is a coboundary, that is when the second structure is a *freezing structure*, the algorithm used to generate a hierarchy of first integrals is known as the *translation argument principle* [3, 25]. Let H_λ be a function on $\mathfrak{g}^*$ which is a Casimir function of the Poisson structure

$$\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_0 + \lambda\{\cdot,\cdot\}_{LP}.$$

That is, for every function F one has

$$\{H_\lambda, F\}_\lambda = 0.$$

Suppose that H_λ can be expressed as a series

$$H_\lambda = H_0 + \lambda H_1 + \lambda^2 H_2 + \cdots$$

Then, one can check that H_0 is a Casimir function of $\{\cdot,\cdot\}_0$ and that for all k, the Hamiltonian vector field of H_{k+1} with respect to $\{\cdot,\cdot\}_0$ coincides with the Hamiltonian vector field of H_k with respect to $\{\cdot,\cdot\}_{LP}$. Furthermore, all the Hamiltonians H_k are in involution with respect to both Poisson structures and the corresponding Hamiltonian vector fields commute with each other. In practice, to obtain such a Casimir function H_λ, one chooses a Casimir function H of the Poisson structure $\{\cdot,\cdot\}_{LP}$ and then *translates the argument*

$$H_\lambda(m) = H(m_0 + \lambda m).$$

The above method has been successfully applied to the KdV equation viewed as a Hamiltonian field on the dual of the Virasoro algebra.

Appendix A. The Gelfand-Fuks Cohomology

Gelfand and Fuks [20, 22] have developed a systematic method to compute the cohomology of the Lie algebra of vector fields on a smooth manifold. This theory is quite sophisticated. The aim of this section is to present a computation of the first two cohomological groups of $\mathrm{Vect}(S^1)$, using only elementary arguments.

The first difficulty when we deal with infinite dimensional Lie algebras like $\mathrm{Vect}(S^1)$ is to define what we call a *cochain*, since a linear or a multilinear map on $\mathrm{Vect}(S^1)$ may be too vague as already stated.

Definition A.1. A p-cochain γ on $\mathrm{Vect}(S^1)$ with values in $\mathbb{R}$ is called *local* if it has the following expression

$$\gamma(u_1, \dots, u_p) = \int_{S^1} P(u_1, \dots, u_p)\, dx$$

where P is a p-linear differential operator.

It is easy to check that if γ is local then $\partial\gamma$ is also local. In the sequel, a cochain on $\mathrm{Vect}(S^1)$ will always mean a *local cochain*[19]. The associated cohomology is called the *Gelfand-Fuks cohomology.*

[19]Using a theorem of Peetre [45], a local cochain can be characterized by the condition

$$\bigcap_{i=1}^{p} Supp(f_i) = \emptyset \Rightarrow \gamma\,(u_1, \dots, u_p) = 0.$$

A.1. **The first cohomology group.** A *local* 1-cochain γ on $\mathrm{Vect}(S^1)$ has the following expression

$$\gamma(u) = \int_{S^1} P(u)\, dx,$$

where P is a linear differential operator. Integrating by parts, we can write it as

$$\gamma(u) = \int_{S^1} mu\, dx,$$

where $m \in C^\infty(S^1)$ is uniquely defined by γ.

Proposition A.2.

$$H^1_{GF}(\mathrm{Vect}(S^1); \mathbb{R}) = \{0\}\,.$$

Proof. If γ is a 1-cocycle, it satisfies the condition

$$\gamma([u,v]) = 0,$$

for all u, v in $\mathrm{Vect}(S^1)$. It a very general result that a Lie algebra which is equal to its commutator algebra has a trivial 1-dimensional cohomology group. Indeed, a linear functional which vanishes on commutators, vanishes everywhere. The proposition is therefore a corollary of lemma 4.1. □

A.2. **The second cohomology group.** A local 2-cochain γ on $\mathrm{Vect}(S^1)$ has the following expression

$$\gamma(u,v) = \int_{S^1} P(u,v)\, dx$$

where P is a quadratic differential operator. Integrating by parts, we can write it as

$$\gamma(u,v) = \int_{S^1} uK(v)\, dx,$$

where $K : C^\infty(S^1) \to C^\infty(S^1)$ is a linear differential operator

$$K = \sum_{k=0}^{n} a_k(x) D^k$$

which is skew-symmetric relatively to the L^2-inner product. This operator is uniquely defined by γ. If moreover γ is a 2-coboundary, there exists $m \in \mathfrak{g}^*$ such that $\gamma = \partial m$, that is

$$\gamma(u,v) = -\int_{S^1} m[u,v]\, dx = \int_{S^1} (ad_u^*\, m) v\, dx,$$

where $u, v \in \mathfrak{g}$. We will therefore introduce the following notation

$$\partial m\,(u) = ad_u^*\, m = mu_x + (mu)_x = 2mu_x + m_x u, \tag{27}$$

to represent the coboundary of the 1-cochain $m \in \mathfrak{g}^*$.

Proposition A.3. *The cohomology group* $H^2_{GF}(\mathrm{Vect}(S^1); \mathbb{R})$ *is one dimensional. It is generated by the* Virasoro cocycle

$$\mathrm{vir}(u,v) = \int_{S^1} (u'v'' - v'u'')\, dx.$$

Proof. Let γ be a 2-cocycle and K the corresponding linear differential operator. The cocycle condition $\partial\gamma = 0$ leads to the following condition on K

$$K([u,v]) = ad_u^* K(v) - ad_v^* K(u), \tag{28}$$

for all $u, v \in C^\infty(S^1)$. Let $w \in C^\infty(S^1)$ with zero integral and $W \in C^\infty(S^1)$ a primitive of w, we have $w = [1, W]$ and hence

$$\begin{aligned} K(w) &= K([1, W]) \\ &= ad_1^* K(W) - ad_W^* K(1) \\ &= K(W)' - (2a_0 W' + a_0' W) \\ &= \left(a_1' w + a_2' w' + \cdots + a_n' w^{(n-1)}\right) + K(w) - 2a_0 w. \end{aligned}$$

Therefore we have

$$(a_1' - 2a_0)w + a_2' w' + \cdots + a_n' w^{(n-1)} = 0$$

for all periodic function w with zero integral which leads to $2a_0 = a_1'$ and $a_k = \text{const.}$, for $2 \le k \le n$. That is, any linear differential linear operator K which satisfies (28) can be written

$$K = \partial m + \sum_{k=2}^{n} \lambda_k D^k,$$

where m is a smooth periodic function[20] and the λ_k are real numbers. Using again equation (28), we get for all periodic functions u, v

$$\sum_{k=2}^{n} \lambda_k (uv' - vu')^{(k)} = 2\sum_{k=2}^{n} \lambda_k (v^{(k)} u' - u^{(k)} v') + \sum_{k=2}^{n} \lambda_k (v^{(k+1)} u - u^{(k+1)} v),$$

which can be rewritten using Leibnitz rule as

$$\sum_{k=2}^{n} \lambda_k \left\{ \sum_{p=1}^{k-1} C_k^p (u^{(p)} v^{(k+1-p)} - v^{(p)} u^{(k+1-p)}) + 3(u^{(k)} v' - v^{(k)} u') \right\} = 0.$$

If we fix v and consider this expression as a linear differential equation in u, all the coefficients of that operator must be zero, and in particular for the coefficient of u' we have

$$\sum_{k=2}^{n} \lambda_k (k-3) v^{(k)} = 0.$$

Therefore we have $\lambda_k = 0$ for $k \neq 3$. Since D^3 is easily seen to verify (28), we can conclude that every cocycle operator K is of the form

$$K = \lambda D^3 + \partial m$$

for some $\lambda \in \mathbb{R}$ and m in $C^\infty(S^1)$. Since every coboundary operator ∂m is a linear differential operator of order 1, D^3 represent a non-trivial cohomology class, which ends the proof. □

[20]Recall that ∂m is the linear differential operator defined by

$$\partial m\,(u) = ad_u^*\, m = mu' + (mu)' = 2mu' + m'u.$$

References

[1] V. I. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(1):319–361, 1966.

[2] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[3] V. I. Arnold and B. A. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[4] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71:1661–1664, 1993.

[5] A. Constantin. On the Cauchy problem for the periodic Camassa-Holm equation. *J. Differential Equations*, 141:218–235, 1997.

[6] A. Constantin. On the inverse spectral problem for the Camassa-Holm equation. *J. Funct. Anal.*, 155(2):352–363, 1998.

[7] A. Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier (Grenoble)*, 50(2):321–362, 2000.

[8] A. Constantin and J. Escher. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. *Comm. Pure Appl. Math.*, 51(5):475–504, 1998.

[9] A. Constantin and J. Escher. On the blow-up rate and the blow-up set of breaking waves for a shallow water equation. *Math. Z.*, 233(1):75–91, 2000.

[10] A. Constantin and B. Kolev. On the geometric approach to the motion of inertial mechanical systems. *J. Phys. A*, 35:R51–R79, 2002.

[11] A. Constantin and B. Kolev. Geodesic flow on the diffeomorphism group of the circle. *Comment. Math. Helv.*, 78(4):787–804, 2003.

[12] A. Constantin and B. Kolev. Integrability of invariant metrics on the diffeomorphism group of the circle. *J. Nonlinear Sci.*, 16(2):109–122, 2006.

[13] A. Constantin, B. Kolev, and J. Lenells. Integrability of invariant metrics on the Virasoro group. *Physics Letters A*, 350(1-2):75–80, 2006.

[14] A. Constantin and H. P. McKean. A shallow water equation on the circle. *Comm. Pure Appl. Math.*, 52:949–982, 1999.

[15] L. D. Faddeev and V. E. Zaharov. The Korteweg-de Vries equation is a fully integrable Hamiltonian system. *Funkcional. Anal. i Priložen.*, 5(4):18–27, 1971.

[16] A. S. Fokas and B. Fuchssteiner. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D*, 4:47–66, 1981/82.

[17] C. S. Gardner. Korteweg-de Vries equation and generalizations. IV. The Korteweg-de Vries equation as a Hamiltonian system. *J. Math. Phys.*, 12:1548–1551, 1971.

[18] I. M. Gel'fand and I. J. Dorfman. Hamiltonian operators and algebraic structures associated with them. *Funktsional. Anal. i Prilozhen.*, 13(4):13–30, 96, 1979.

[19] I. M. Gel'fand and I. J. Dorfman. Hamiltonian operators and infinite-dimensional Lie algebras. *Funktsional. Anal. i Prilozhen.*, 15(3):23–40, 1981.

[20] I. M. Gelfand and D. B. Fuks. Cohomologies of the Lie algebra of vector fields on the circle. *Funkcional. Anal. i Prilovzen.*, 2(4):92–93, 1968.

[21] F. Gesztesy and H. Holden. Algebro-geometric solutions of the Camassa-Holm hierarchy. *Rev. Mat. Iberoamericana*, 19(1):73–142, 2003.

[22] L. Guieu and C. Roger. *Algèbre de Virasoro: aspects géométriques et algébriques.* Soc. Math. France, 2005.

[23] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)*, 7:65–222, 1982.

[24] R. Jost. Poisson brackets. *Reviews Modern Physics*, 36:572–579, 1964.

[25] B. Khesin and G. Misiolek. Euler equations on homogeneous spaces and Virasoro orbits. *Adv. Math.*, 176:116–144, 2003.

[26] B. Kolev. Lie groups and mechanics: an introduction. *J. Nonlinear Math. Phys.*, 11:480–498, 2004.

[27] D. J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.*, 39:422–443, 1895.

[28] Y. Kosmann-Schwarzbach and F. Magri. Poisson-Nijenhuis structures. *Ann. Inst. H. Poincaré Phys. Théor.*, 53(1):35–81, 1990.
[29] Y. Kosmann-Schwarzbach and F. Magri. Lax-Nijenhuis operators for integrable systems. *J. Math. Phys.*, 37(12):6173–6197, 1996.
[30] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.
[31] P. D. Lax. Almost periodic solutions of the KdV equation. *SIAM Rev.*, 18(3):351–375, 1976.
[32] J. Lenells. The correspondence between KdV and Camassa-Holm. *Int. Math. Res. Not.*, (71):3797–3811, 2004.
[33] J. Lenells. Conservation laws of the Camassa-Holm equation. *J. Phys. A*, 38(4):869–880, 2005.
[34] A. Lichnerowicz. Les variétés de Poisson et leurs algèbres de Lie associées. *J. Differential Geometry*, 12(2):253–300, 1977.
[35] E. Loubet. About the explicit characterization of Hamiltonians of the Camassa-Holm hierarchy. *J. Nonlinear Math. Phys.*, 12(1):135–143, 2005.
[36] F. Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19(5):1156–1162, 1978.
[37] H. P. McKean. Integrable systems and algebraic curves. In *Global analysis*, pages 83–200, Berlin, 1979. Springer Lecture Notes in Math., 755.
[38] H. P. McKean. Compatible brackets in Hamiltonian mechanics. In *Important developments in soliton theory*, Springer Ser. Nonlinear Dynam., pages 344–354. Springer, Berlin, 1993.
[39] H. P. McKean. Breakdown of the Camassa-Holm equation. *Comm. Pure Appl. Math.*, 57(3):416–418, 2004.
[40] J. Milnor. Remarks on infinite-dimensional Lie groups. In *Relativity, Groups and Topology*, pages 1009–1057, Amsterdam, 1984. North-Holland.
[41] G. Misiołek. A shallow water equation as a geodesic flow on the Bott-Virasoro group. *J. Geom. Phys.*, 24(3):203–208, 1998.
[42] J. J. Moreau. Une méthode de "cinématique fonctionnelle" en hydrodynamique. *C. R. Acad. Sci. Paris*, 249:2156–2158, 1959.
[43] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.
[44] V. Y. Ovsienko and B. A. Khesin. The super Korteweg-de Vries equation as an Euler equation. *Funktsional. Anal. i Prilozhen.*, 21(4):81–82, 1987.
[45] J. Peetre. Une caractérisation abstraite des opérateurs différentiels. *Math. Scand.*, 7:211–218, 1959.
[46] J. Praught and R. G. Smirnov. Andrew Lenard: a mystery unraveled. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 1:Paper 005, 7 pp. (electronic), 2005.
[47] J. M. Souriau. *Structure of Dynamical Systems*. Birkhäuser Boston Inc., Boston, MA, 1997.
[48] T. Tao. Low-regularity global solutions to nonlinear dispersive equations. In *Surveys in analysis and operator theory (Canberra, 2001)*, volume 40 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 19–48. Austral. Nat. Univ., Canberra, 2002.
[49] I. Vaisman. *Lectures on the geometry of Poisson manifolds*. Birkhäuser Verlag, Basel, 1994.

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